MATH2230 Midterm 1 solution

1a)(20 marks) Method 1:

Let $z = x + yi \neq 0$, we have $\dfrac{1}{z} = \dfrac{1}{x+yi} = \dfrac{z-yi}{x^2+y^2}$

If $\text{Re}\left(\dfrac{1}{z}\right) = \dfrac{1}{2}$, then

$$\begin{aligned} \frac{x}{x^2+y^2} &= \frac{1}{2} \\ 2x &= x^2+y^2 \\ (x-1)^2+y^2 &= 1 \end{aligned}$$

Hence $\left\{z \in \mathbb{C} : Re\left(\dfrac{1}{z}\right) = \dfrac{1}{2}\right\}$ contains the points on the circle $\{z \in \mathbb{C} : |z-1| = 1\} \setminus \{0\}$.

Method 2:

$$\begin{aligned} Re\left(\frac{1}{z}\right) &= \frac{1}{2} \\ Re\left(\frac{1}{z} - \frac{1}{2}\right) &= 0 \\ Im\left(i\left(\frac{1}{z} - \frac{1}{2}\right)\right) &= 0 \\ Im\left((-i)\left(\frac{z-2}{z-0}\right)\right) &= 0 \end{aligned}$$

Compare with $\text{Im}\left(\dfrac{\frac{z-z_1}{z-z_2}}{\frac{z_3-z_1}{z_3-z_2}}\right) = 0$, we have the same result.

1b)(20 marks) Method 1:

Let $z = x + yi$, we have $\dfrac{\overline{z}+1-3i}{4-i} = \dfrac{[(x+1)-(3+y)i](4+i)}{17}$

If $\text{Im}\left(\dfrac{\overline{z}+1-3i}{4-i}\right) > 1$, then

$$\begin{aligned} \frac{x+1-4(3+y)}{17} &> 1 \\ x-4y &> 28 \end{aligned}$$

It contains all the points strictly lie on the right hand side of the line $x - 4y = 28$.

Method 2:

$$\begin{aligned} Im\left(\frac{\overline{z}+1-3i}{4-i}\right) &> 1 \\ \overline{Im\left(\frac{z+1+3i}{4+i}\right)} &> 1 \\ Im\left(\frac{z+1+3i}{4+i}\right) &< -1 \\ Im\left(\frac{z+7i}{4-6i+7i}\right) &< 0 \end{aligned}$$

It contains all the points strictly lie on the right hand side of the line passing through $z_1 = -7i$ and $z_2 = 4 - 6i$.

2)(15 marks)

$$\begin{aligned} 8+8\sqrt{3}i &= 16e^{\pi i/3} \\ z^4 &= 16e^{\pi i/3+2ki\pi} \text{ for } k\in\mathbb{Z} \\ z &= 2e^{\frac{\pi i}{12}+\frac{k\pi i}{2}} \text{ for } k=0,1,2,3 \end{aligned}$$

3a)(5 marks) The derivative of f at $z=z_0$ is defined to be the limit $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ (or $\lim_{z\to z_0}\frac{f(z_0+h)-f(z_0)}{h}$) if it exists.

3b)(25 marks) Case 1 : If $z_0=0$, then we choose $h=\delta>0$,

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|\delta|}{\delta}=1$$

Choosing $h=-\delta<0$,

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|\delta|}{-\delta}=-1$$

f cannot be derivable at $z_0=0$.

Case 2: If $z_0\neq 0$, then we choose $h=\delta>0$ and $z_0=x+yi$,

$$\begin{aligned} \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &= \lim_{\delta\to 0}\frac{\sqrt{(x+\delta)^2+y^2}-\sqrt{x^2+y^2}}{\delta} \\ &= \lim_{\delta\to 0}\frac{(x+\delta)^2+y^2-(x^2+y^2)}{\delta\sqrt{(x+\delta)^2+y^2}+\sqrt{x^2+y^2}} \\ &= \lim_{\delta\to 0}\frac{2\delta x+\delta^2}{\delta\sqrt{(x+\delta)^2+y^2}+\sqrt{x^2+y^2}}=\frac{x}{|z|} \end{aligned}$$

Choosing $h=i\delta$ for $\delta>0$, we have

$$\begin{aligned} \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &= \lim_{\delta\to 0}\frac{\sqrt{x^2+(y+\delta)^2}-\sqrt{x^2+y^2}}{\delta i} \\ &= \lim_{\delta\to 0}\frac{2\delta y+\delta^2}{\delta i\sqrt{x^2+(y+\delta)^2}+\sqrt{x^2+y^2}}=\frac{-iy}{|z|} \end{aligned}$$

Hence, it is not derivable.

4)(15 marks) Since the principal value of power function is not continuous on the principal branch $=\{arg(z)=-\pi\}\bigcup\{z=0\}$. Thus $f(iz-2)$ is not continuous on $\{arg(iz-2)=-\pi\}\bigcup\{iz-2=0\}$. If $\arg(iz-2)=-\pi$, then $iz-2=r<0$. If $z=x+iy$, then

$$\begin{aligned} ix-y-2 &= r \\ (-y-2-r)+ix &= 0 \end{aligned}$$

We conclude that $-y-2=r<0$ and $x=0$, hence g is not continuous on $\{z=y:y\geq -2\}$.